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Complex Angular Momentum in Spinor Bethe-Salpeter Equation*

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A model of fermion-antifermion scattering, mediated by pseudoscalar neutral bosons, is described by the corresponding spinorial Bethe-Salpeter equation in the ladder approximation. The decomposition of the equation into partial waves by means of fusion amplitudes and conventional spherical harmonics is discussed in detail; various important symmetries of the resulting kernel and Born amplitudes are pointed out. The resulting set of coupled equations is continued into the complex angular momentum (J) plane, and it is shown that Fredholm theory is inapplicable for any J. The equations are solved in a weak coupling, low-energy limit by an iterative scheme. The resulting solutions exhibit a cut of the square-root type extending along the real axis to $G/2\pi$ (G=coupling constant) in the right-hand plane; other cuts and poles prevent the extension of our solutions into the left-hand J plane. The dominance of the cut is used to extract the large momentum transfer limit and obtain certain results for the high-energy limit in the cross channel. The total cross section for fermion-antifermion annihilation is extracted by means of the optical theorem and is found to exhibit an energy dependence of the form $t^{(G/2\pi)-1}[\ln t]^{-3/2}$, where t is the c.m. energy squared.

1. INTRODUCTION AND SUMMARY

In the present paper we wish to examine the analyticity in the complex angular momentum (J) of the Bethe-Salpeter amplitude in the ladder approximation.[1] The case of two scalar particles scattering via exchange of a scalar particle has been previously investigated[2]; we shall consider here the scattering of a spinor particle and its antiparticle (for the sake of convenience, we shall call them "nucleon" and "antinucleon" of mass m) exchanging pseudoscalar bosons[3] ("pion" of mass μ).

Aside from the intrinsic interest in the Regge poles that might be associated with the "nucleonium" states and resonances, the analyticity in the complex angular momentum of the ladder amplitude will tell us about the high-energy behavior of the nucleon-antinucleon annihilation into multiple pions in a model schematically summarized in Fig. 1. To elaborate, the leading singularity that lies to the rightmost in the complex J plane will control the high-energy behavior of the nucleon-antinucleon scattering in the crossed channel [Fig. 1(b)] and the imaginary part of the crossed amplitude at zero momentum transfer gives the cross section for the $N\bar{N}$ annihilation into multiple bosons.[4]

The partial wave Bethe-Salpeter equation for two spinor particles is highly singular, so that the method used in Ref. 2 is no longer applicable. In fact, the kernel of the integral equation is not square integrable, and the usual Fredholm theory[5] has nothing to say about the nature of the solutions to this equation. (See Appendix II.) We shall show, however, that, for $|E|\ll m$ ($2E=\sqrt{s}$ is the total c.m. energy of the system) and in the weak coupling limit, the iterative solution converges to the right of the line Re$J=0$, excluding a small region around $J=0$; the most singular parts (near the region around $J=0$) of the iterative series is summed by the technique developed by Sawyer[6] and continued into the region. We find that, for $|E|\ll m$, the amplitude is analytic in J in the half-plane Re$J>0$ except for a cut along the real J axis and embedded within the region.

The high-energy behavior of the scattering amplitude in the crossed channel is therefore controlled by this "Regge cut" and goes as (t is the total c.m. energy squared in the crossed channel)

$$(-t)^{\alpha}/[\ln(-t)]^{3/2},$$

characteristic of a Regge cut of the square root type.[6,7]

Bjorken and Wu,[8] and Sawyer[6] have shown several examples in which the leading singularities in the complex J plane are cuts. These cases correspond, more or less, to the r^{-2} type potential in nonrelativistic scattering. In our case, the Regge cut arises from the spin effect: In the nonrelativistic limit the interaction

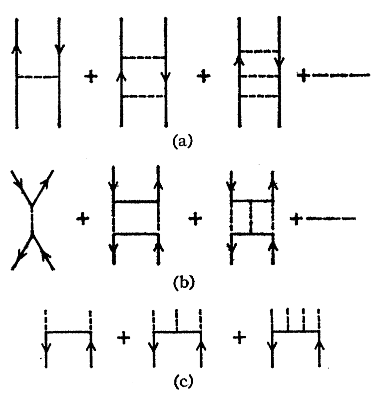

FIG. 1. (a) The ladder diagrams in the s channel. (b) The annihilation-recreation scattering diagrams in the crossed channel. (c) $N\bar{N}$ annihilation into mesons.

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[1] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951); G. C. Wick, *ibid.* **96**, 1124 (1952).
[2] B. W. Lee and R. F. Sawyer, Phys. Rev. **127**, 2266 (1962).
[3] We do not pretend that our consideration is based on anything but a model.
[4] D. Amati, A. Stanghellini, and S. Fubini, Nuovo Cimento **24**, 896 (1962).
[5] For instance, F. Smithies, *Integral Equations* (Cambridge University Press, Cambridge, England, 1958).
[6] R. F. Sawyer (to be published).
[7] P. G. O. Freund and R. Oehme, Phys. Rev. Letters **10**, 199 (1963).
[8] J. D. Bjorken and T. T. Wu (to be published).

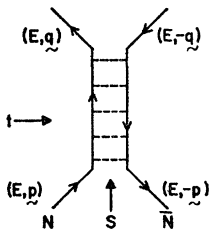

FIG. 2. The kinematics of $N\bar{N}$ scattering in the s channel.

we consider includes singular potentials, such as spin-orbit coupling and tensor forces, which are at least as singular as the r^{-2} potential.

In the next section we decompose the Bethe-Salpeter equation into partial waves. The method used is very similar to that of Gourdin,[9] but differs from his in that we expand the amplitude in the usual spherical harmonics rather than in the hyperspherical harmonics. Our method here is closely related to that used in Ref. 2 and provides an unambiguous and direct way of extending the equation to complex values of J.

In Sec. 3 we decouple the various integral equations by continuing them in the angular momentum near $J=0$. The resulting equations are solved by iteration. Several differences between our work and that of Sawyer are pointed out. Section 4 is concerned with obtaining a relation between the fusion amplitudes of Gourdin and the helicity amplitudes of Jacob and Wick.[10,11] These are, in turn, related to a set of amplitudes on which can be performed the Sommerfeld-Watson transformation[11,12] In Sec. 5 we relate the amplitudes of Sec. 4 to a set of invariant amplitudes suitable for continuing into the crossed channel; and in Sec. 6 we examine the high-energy behavior in this channel. We obtain both a differential cross section for forward scattering, and a total cross section for nucleon-antinucleon annihilation. The Appendixes contain several useful mathematical formulas and certain pertinent mathematical results.

2. PARTIAL-WAVE BETHE-SALPETER EQUATION

We consider the scattering of a nucleon-antinucleon system in the s channel where the scattering occurs only through the exchange of neutral pseudoscalar mesons. We take the ps coupling between the mesons and nucleons. The momenta of the initial and final particles are defined as shown in Fig. 2. In the ladder approximation,[13] the transition amplitude R may be written as

$$R=\frac{1}{(2\pi)^3}\frac{m}{E}\bar{u}_{\lambda_1'}(\mathbf{q})\gamma_5\Psi^{\lambda_1\lambda_2}(\mathbf{q},\mathbf{p})\gamma_5 v_{-\lambda_2'}(\mathbf{q})\,,\qquad(1)$$

where λ_1, λ_2, λ_1', λ_2' denote the helicities of the initial and final nucleons, and antinucleons, respectively. The amplitude $\Psi^{\lambda_1\lambda_2}(\mathbf{q},\mathbf{p})$ is related to the Bethe-Salpeter (BS) amplitude by

$$\Psi^{\lambda_1\lambda_2}(\mathbf{q},\mathbf{p})=\Psi^{\lambda_1\lambda_2}(\mathbf{q},\,q_0=0)$$

and $\Psi^{\lambda_1\lambda_2}(q)$ satisfies the BS equation[9,14]

$$\gamma_0=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \boldsymbol{\gamma}=\begin{pmatrix}0&\boldsymbol{\sigma}\\-\boldsymbol{\sigma}&0\end{pmatrix},\quad \gamma_5=-i\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

$$\Psi^{\lambda_1\lambda_2}(q)=\Psi_B{}^{\lambda_1\lambda_2}(q)+\frac{iG^2}{(2\pi)^4}\int\frac{d^4k}{[(k-q)^2-\mu^2]}\frac{\gamma\cdot(k+E)+m}{[(k+E)^2-m^2]}\times\gamma_5\Psi^{\lambda_1\lambda_2}(k)\gamma_5\frac{\gamma\cdot(k-E)+m}{[(k-E)^2-m^2]},\qquad(2)$$

where $2E=\sqrt{s}=2(p^2+m^2)^{1/2}$ is the total c.m. energy of the system, and the zeroth component of q, q_0, is the so-called relative energy.[1] The Born term Ψ_B is given by

$$\Psi_B{}^{\lambda_1\lambda_2}(q)=-2\pi iG^2\frac{m}{E}\frac{u_{\lambda_1}(\mathbf{p})\bar{v}_{-\lambda_2}(\mathbf{q})}{(p-q)^2-\mu^2}.\qquad(3)$$

Following the method described by Gourdin,[9] we decompose Ψ into four 2×2 Pauli spaces:

$$\Psi=\begin{pmatrix}\frac{1}{2}(S+T)+\frac{1}{2}(\mathbf{V}+\mathbf{U})\cdot\boldsymbol{\sigma}, & \frac{1}{2}(B+C)+\frac{1}{2}(\mathbf{F}+\mathbf{G})\cdot\boldsymbol{\sigma}\\ \frac{1}{2}(B-C)+\frac{1}{2}(\mathbf{F}-\mathbf{G})\cdot\boldsymbol{\sigma}, & \frac{1}{2}(S+T)+\frac{1}{2}(\mathbf{V}-\mathbf{U})\cdot\boldsymbol{\sigma}\end{pmatrix}.\qquad(4)$$

Substituting Ψ into the BS equation, we find the following set of coupled integral equations for S, T, etc.:

$$\begin{aligned}
S&=S_B+K\{(m^2+k_0^2-E^2-k^2)S-2mk_0T+2E\mathbf{k}\cdot\mathbf{F}-2m\mathbf{k}\cdot\mathbf{G}\}\,,\\
T&=T_B+K\{-(m^2+k_0^2-E^2+k^2)T+2mk_0S-2k_0\mathbf{k}\cdot\mathbf{G}\}\,,\\
\mathbf{V}&=\mathbf{V}_B+K\{(m^2+k_0^2-E^2+k^2)\mathbf{V}-2mk_0\mathbf{U}-2\mathbf{k}(\mathbf{k}\cdot\mathbf{V})+2E\mathbf{k}B-2m\mathbf{k}C-2ik_0\mathbf{k}\times\mathbf{F}\}\,,\\
\mathbf{U}&=\mathbf{U}_B+K\{-(m^2+k_0^2-E^2-k^2)\mathbf{U}+2mk_0\mathbf{V}-2\mathbf{k}(\mathbf{k}\cdot\mathbf{U})-2k_0\mathbf{k}C-2im\mathbf{k}\times\mathbf{F}+2ik_0\mathbf{k}\times\mathbf{G}\}\,,\qquad(5)\\
B&=B_B+K\{(m^2+E^2-k_0^2+k^2)B-2mEC-2E\mathbf{k}\cdot\mathbf{V}\}\,,\\
C&=C_B+K\{-(m^2+E^2-k_0^2-k^2)C+2mEB-2m\mathbf{k}\cdot\mathbf{V}+2k_0\mathbf{k}\cdot\mathbf{U}\}\,,\\
\mathbf{F}&=\mathbf{F}_B+K\{(m^2+E^2-k^2-k_0^2)\mathbf{F}-2mE\mathbf{G}-2E\mathbf{k}S+2\mathbf{k}(\mathbf{k}\cdot\mathbf{F})+2ik_0\mathbf{k}\times\mathbf{V}-2im\mathbf{k}\times\mathbf{U}\}\,,\\
\mathbf{G}&=\mathbf{G}_B+K\{-(m^2+E^2-k_0^2+k^2)\mathbf{G}+2mE\mathbf{F}+2\mathbf{k}(\mathbf{k}\cdot\mathbf{G})-2m\mathbf{k}S+2k_0\mathbf{k}T-2iE\mathbf{k}\times\mathbf{V}\}\,,
\end{aligned}$$

[9] M. Gourdin, thesis, Université de Paris, 1958 (unpublished); Nuovo Cimento 7, 338 (1958); Ann. Phys. (Paris) 4, 641 (1959).

[10] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) 7, 404 (1959).

[11] M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. 120, 2250 (1960).

[12] V. B. Berestetsky, Phys. Letters 3, 175 (1963).

[13] We point out here that we are neglecting annihilation diagrams in the s channel. The inclusion of them might well alter our results, but would also lead to great mathematical difficulties.

[14] We use the metric $g_{00}=-g_{ii}=1$ $(i=1,2,3)$, and the γ matrices.

where K stands for the operation

$$K\{S\}=\frac{-iG^2}{(2\pi)^4}\times\int\frac{d^4kS}{[(k-q)^2-\mu^2][(k+E)^2-m^2][(k-E)^2-m^2]}.$$

Next we decompose the above equations by expanding the various fusion amplitudes in spherical harmonics.

$$S=\sum_{J,M}S_{J,M}Y_{J,M}(\theta,\varphi)\,,\quad\cdots;$$
$$\mathbf{V}=\sum_{J,M}\{V_{J,M}{}^{+}\mathbf{Y}_{J,J+1,M}+V_{J,M}{}^{0}\mathbf{Y}_{J,J,M}+V_{J,M}{}^{-}\mathbf{Y}_{J,J-1,M}\}\,,\quad\cdots,\qquad(6)$$

where the $\mathbf{Y}_{JLM}$ are the vector spherical harmonics discussed by Edmonds.[15] The z axis is taken to be the direction of the incident momentum. Several relations which are very useful for performing this reduction are listed in Appendix I. Furthermore, the boson propagator can be expanded as

$$\frac{1}{(k-q)^2-\mu^2}=-\frac{2\pi}{qk}\sum_{l,M}Y_{l,M}{}^{*}(\Omega_K)Y_{l,M}(\Omega_q)Q_l(y)\,,$$

where

$$y=[k^2+q^2+\mu^2-(k_0-q_0)^2]/2kq\,,$$

and $Q_J(y)$ is the Legendre function of the second kind.

We find that the resulting set of coupled equations can be written in the form

$$\Psi_j{}^J=\Psi_{Bj}{}^J+\frac{iG^2}{(2\pi)^3q}\sum_i\int\frac{kdkdk_0M_{ji}{}^J(q\cdot k)\Psi_i{}^J(k)}{[(k+E)^2-m^2][(k-E)^2-m^2]}\,,\qquad(7)$$

where $\Psi_i{}^J$ stands for the column matrix S_J, T_J, $\cdots$, $V_J{}^+$, $V_J{}^-$, M_{ij} is a 16×16 matrix which is written in block diagonal form in terms of two 8×8 matrices $M^{(1)}$, $M^{(2)}$. Corresponding to $M^{(1)}$ we have the column matrix

$$\{S_J,T_J,V_J{}^0,U_J{}^0,F_J{}^+,F_J{}^-,G_J{}^+,G_J{}^-\}$$

and to $M^{(2)}$

$$\{B_J,C_J,F_J{}^0,G_J{}^0,V_J{}^+,V_J{}^-,U_J{}^+,U_J{}^-\}.$$

The elements of the matrices $M^{(1)}$ and $M^{(2)}$ are tabulated in Table I.

One ought to note at this point that the coupled set of equations represented by Table I is valid for $J\geq1$. In the next section we shall continue the equations in J down to J close to zero. However, the physical solution for $J=0$ cannot be obtained from the above set, but rather comes from the set obtained by setting $Q_{J-1}(y)\equiv0$.

The breakup of the integral equation into two disjoint parts is a direct consequence of parity conservation. Equation (2) is invariant under the parity operation $\mathcal{P}$:

$$\Psi(\mathbf{q},q_0)\rightarrow\mathcal{P}\{\Psi(\mathbf{q},q_0)\}\equiv\gamma_0\Psi(-\mathbf{q},q_0)\gamma_0\,.\qquad(8)$$

Hence, if we write $\Psi(q)$ as a linear combination of amplitudes which are even and odd under the operation $\mathcal{P}$, we see that the two parts are decoupled. Substituting Eq. (6) into Eq. (4) and performing the operation indicated in Eq. (8), we find that the first eight Ψ_i's, S_J, T_J, $\cdots$, $G_J{}^+$, $G_J{}^-$, belong to the states of parity $(-1)^{J+1}$ (singlet states and triplet states with $J=L$), while the rest belong to the states of parity $(-1)^J$ (two linear combinations of triplet states with $L=J\pm1$ for each J).

We next compute the Born terms Ψ_{Bi}. These terms, which depend on the initial helicity states, can be formed from a consideration of the outer product of two spinors.

$$\Phi_{\lambda_1\lambda_2}=u_{\lambda_1}(\mathbf{p})\otimes\bar{v}_{-\lambda_2}(\mathbf{p})=\left(\frac{p}{2m}\right)\begin{pmatrix}\chi_{\lambda_1}\\ [2\lambda_1p/(E+m)]\chi_{\lambda_1}\end{pmatrix}\left(\chi_{\lambda_2}{}^{\dagger},\ \frac{2\lambda_2p\chi_{\lambda_2}{}^{\dagger}}{E-m}\right);\qquad(9)$$

$u(\mathbf{p})$, $v(\mathbf{p})$ satisfy the equations $(\gamma\cdot p-m)u(\mathbf{p})=(\gamma\cdot p+m)v(\mathbf{p})=0$. Also $\sigma_3\chi_\lambda=2\lambda\chi_\lambda$. For a discussion of the proper combinations of spinors to be used consult Ref. 10 or 11. The outer product $\chi_{\lambda_1}\chi_{\lambda_2}{}^\dagger$ can be written as a sum of Pauli matrices according to the values of λ_1, λ_2:

$2\lambda_1$, $2\lambda_2$	$\chi_{\lambda_1}\chi_{\lambda_2}{}^\dagger$
1, 1	$\frac{1}{2}(1+\hat{p}\cdot\boldsymbol{\sigma})$
1, −1	$\frac{1}{2}(\hat{e}_1+i\hat{e}_2)\cdot\boldsymbol{\sigma}$
−1, 1	$\frac{1}{2}(\hat{e}_1-i\hat{e}_2)\cdot\boldsymbol{\sigma}$
−1, −1	$\frac{1}{2}(1-\hat{p}\cdot\boldsymbol{\sigma})$. (10)

Then, using the fact that

$$\Psi_B{}^{\lambda_1\lambda_2}=-2\pi iG^2\frac{m}{E}\left(-\frac{2\pi}{pq}\sum_{l,m}Q_l(y)\times\left(\frac{2l+1}{4\pi}\right)^{1/2}Y_{l0}(\Omega_q)\delta_{M0}\right)\Phi_{\lambda_1\lambda_2}\qquad(11)$$

and the relations given in Appendix I, one can read off the terms of $\Psi_B{}^{\lambda_1\lambda_2}$. The results are summarized in Table II.

Again we point out that the $J=0$ Born terms are obtained by setting $Q_{J-1}(y)=0$, as well as $J=0$. Since we shall not need to consider these $J=0$ terms, we drop any further mention of them. The Born terms as written can be continued to within an ϵ of $J=0$.

One should note from Table II that if we form the singlet and triplet combinations of the eight amplitudes

[15] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957), p. 81.

TABLE I. The elements of the matrices $M_{ij}^{(1)}$ and $M_{ij}^{(2)}$.

j	1	2	3	4 ($\langle M_{ij}^{(1)}\rangle$)	5	6	7	8
M_{1j}/Q_J	N_{11}	$-2mk_0$	$\cdots$	$\cdots$	$-2Ek\left(\frac{J+1}{2J+1}\right)^{1/2}$	$2Ek\left(\frac{J}{2J+1}\right)^{1/2}$	$2mk\left(\frac{J+1}{2J+1}\right)^{1/2}$	$-2mk\left(\frac{J}{2J+1}\right)^{1/2}$
M_{2j}/Q_J	$2mk_0$	$-N_{33}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$2kk_0\left(\frac{J+1}{2J+1}\right)^{1/2}$	$-2kk_0\left(\frac{J}{2J+1}\right)^{1/2}$
M_{3j}/Q_J	$\cdots$	$\cdots$	N_{33}	$-2mk_0$	$2kk_0\left(\frac{J}{2J+1}\right)^{1/2}$	$2kk_0\left(\frac{J+1}{2J+1}\right)^{1/2}$	$\cdots$	$\cdots$
M_{4j}/Q_J	$\cdots$	$\cdots$	$2mk_0$	$-N_{11}$	$2mk\left(\frac{J}{2J+1}\right)^{1/2}$	$2mk\left(\frac{J+1}{2J+1}\right)^{1/2}$	$-2Ek\left(\frac{J}{2J+1}\right)^{1/2}$	$-2Ek\left(\frac{J+1}{2J+1}\right)^{1/2}$
M_{5j}/Q_{J+1}	$2Ek\left(\frac{J+1}{2J+1}\right)^{1/2}$	$\cdots$	$-2kk_0\left(\frac{J}{2J+1}\right)^{1/2}$	$2mk\left(\frac{J}{2J+1}\right)^{1/2}$	N_{55}	$-2k^2\frac{[J(J+1)]^{1/2}}{2J+1}$	$-2mE$	$\cdots$
M_{6j}/Q_{J-1}	$-2Ek\left(\frac{J}{2J+1}\right)^{1/2}$	$\cdots$	$-2kk_0\left(\frac{J+1}{2J+1}\right)^{1/2}$	$2mk\left(\frac{J+1}{2J+1}\right)^{1/2}$	$-2k^2\frac{[J(J+1)]^{1/2}}{2J+1}$	N_{66}	$\cdots$	$\cdots$
M_{7j}/Q_{J+1}	$2mk\left(\frac{J+1}{2J+1}\right)^{1/2}$	$-2kk_0\left(\frac{J+1}{2J+1}\right)^{1/2}$	$\cdots$	$2Ek\left(\frac{J}{2J+1}\right)^{1/2}$	$2mE$	$\cdots$	$-N_{66}$	$-2k^2\frac{[J(J+1)]^{1/2}}{2J+1}$
M_{8j}/Q_{J-1}	$-2mk\left(\frac{J}{2J+1}\right)^{1/2}$	$2kk_0\left(\frac{J}{2J+1}\right)^{1/2}$	$\cdots$	$2Ek\left(\frac{J+1}{2J+1}\right)^{1/2}$	$\cdots$	$2mE$	$-2k^2\frac{[J(J+1)]^{1/2}}{2J+1}$	$-N_{55}$
		$N_{11}=m^2+k_0^2-E^2-k^2$	$N_{33}=m^2+k_0^2-E^2+k^2$	$N_{55}=m^2+E^2-k_0^2+k^2/(2J+1)$		$N_{66}=m^2+E^2-k_0^2-k^2/(2J+1)$		

j	9	10	11	12 ($\langle M_{ij}^{(2)}\rangle$)	13	14	15	16
M_{9j}/Q_J	N_{99}	$-2mE$	$\cdots$	$\cdots$	$2Ek\left(\frac{J+1}{2J+1}\right)^{1/2}$	$-2Ek\left(\frac{J}{2J+1}\right)^{1/2}$	$\cdots$	$\cdots$
M_{10j}/Q_i	$2mE$	$-N_{1111}$	$\cdots$	$\cdots$	$2mk\left(\frac{J+1}{2J+1}\right)^{1/2}$	$-2mk\left(\frac{J}{2J+1}\right)^{1/2}$	$-2kk_0\left(\frac{J+1}{2J+1}\right)^{1/2}$	$2kk_0\left(\frac{J}{2J+1}\right)^{1/2}$
M_{11j}/Q_J	$\cdots$	$\cdots$	N_{1111}	$-2mE$	$-2kk_0\left(\frac{J+1}{2J+1}\right)^{1/2}$	$-2kk_0\left(\frac{J}{2J+1}\right)^{1/2}$	$2mk\left(\frac{J+1}{2J+1}\right)^{1/2}$	$2mk\left(\frac{J}{2J+1}\right)^{1/2}$
M_{12j}/Q_i	$\cdots$	$\cdots$	$2mE$	$-N_{99}$	$\cdots$	$\cdots$	$2Ek\left(\frac{J}{2J+1}\right)^{1/2}$	$2Ek\left(\frac{J+1}{2J+1}\right)^{1/2}$
M_{13j}/Q_{J+1}	$-2Ek\left(\frac{J+1}{2J+1}\right)^{1/2}$	$2mk\left(\frac{J+1}{2J+1}\right)^{1/2}$	$2kk_0\left(\frac{J}{J}\right)^{1/2}$	$\cdots$	N_{1313}	$2k^2\frac{[J(J+1)]^{1/2}}{2J+1}$	$-2mk_0$	$\cdots$
M_{14j}/Q_{i-1}	$2Ek\left(\frac{J}{2J+1}\right)^{1/2}$	$-2mk\left(\frac{J}{2J+1}\right)^{1/2}$	$2kk_0\left(\frac{J+1}{2J+1}\right)^{1/2}$	$\cdots$	$2k^2\frac{[J(J+1)]^{1/2}}{2J+1}$	N_{1414}	$\cdots$	$-2mk_0$
M_{15j}/Q_{J+1}	$\cdots$	$2kk_0\left(\frac{J+1}{2J+1}\right)^{1/2}$	$2mk\left(\frac{J}{2J+1}\right)^{1/2}$	$2Ek\left(\frac{J}{2J+1}\right)^{1/2}$	$2mk_c$	$\cdots$	$-N_{1414}$	$2k^2\frac{[J(J+1)]^{1/2}}{2J+1}$
M_{16j}/Q_{J-1}	$\cdots$	$-2kk_0\left(\frac{J}{2J+1}\right)^{1/2}$	$2mk\left(\frac{J+1}{2J+1}\right)^{1/2}$	$-2Ek\left(\frac{J+1}{2J+1}\right)^{1/2}$	$\cdots$	$2mk_0$	$2k^2\frac{[J(J+1)]^{1/2}}{2J+1}$	$-N_{1313}$
		$N_{99}=m^2+E^2+k^2-k_0^2$	$N_{1111}=m^2+E^2-k^2-k_0^2$	$N_{1313}=m^2+k_0^2-E^2-k^2/(2J+1)$		$N_{1414}=m^2+k_0^2-E^2+k^2/(2J+1)$		

TABLE II. The Born term Ψ_{Bi} for the various initial helicities.

	$+\frac{1}{2}, +\frac{1}{2}$	$-\frac{1}{2}, -\frac{1}{2}$	$+\frac{1}{2}, -\frac{1}{2}$	$-\frac{1}{2}, +\frac{1}{2}$
S_{JM}/Q_J	$\frac{iG^2(2\pi)^2}{2Eq}\left(\frac{2J+1}{4\pi}\right)^{1/2}\delta_{M0}$	$+$	$\cdots$	
T_{JM}/Q_J	$\cdots$		$\cdots$	
$V_{JM}{}^0/Q_J$	$\cdots$		$\cdots$	
$U_{JM}{}^0/Q_J$	$\cdots$		$\frac{+iG^2(2\pi)^2}{2Eq}\left(\frac{2J+1}{4\pi}\right)^{1/2}\delta_{M1}$	$+\delta_{M-1}$
$F_{JM}{}^+/Q_{J+1}$	$\frac{-iG^2(2\pi)^2}{2pq}\left(\frac{J+1}{4\pi}\right)^{1/2}\delta_{M0}$	$-$	$\frac{+iG^2(2\pi)^2}{2Eq}\frac{m}{p}\left(\frac{J}{4\pi}\right)^{1/2}\delta_{M1}$	$+\delta_{M-1}$
$F_{JM}{}^-/Q_{J-1}$	$\frac{+iG^2(2\pi)^2}{2pq}\left(\frac{J}{4\pi}\right)^{1/2}\delta_{M0}$	$+$	$\frac{+iG^2(2\pi)^2}{2Eq}\frac{m}{p}\left(\frac{J+1}{4\pi}\right)^{1/2}\delta_{M1}$	$+\delta_{M-1}$
$G_{JM}{}^+/Q_{J+1}$	$\frac{-iG^2(2\pi)^2}{2Eq}\frac{m}{p}\left(\frac{J+1}{4\pi}\right)^{1/2}\delta_{M0}$	$-$	$\frac{+iG^2(2\pi)^2}{2pq}\left(\frac{J}{4\pi}\right)^{1/2}\delta_{M1}$	$+\delta_{M-1}$
$G_{JM}{}^-/Q_{J-1}$	$\frac{+iG^2(2\pi)^2}{2Eq}\frac{m}{p}\left(\frac{J}{4\pi}\right)^{1/2}\delta_{M0}$	$+$	$\frac{+iG^2(2\pi)^2}{2pq}\left(\frac{J+1}{4\pi}\right)^{1/2}\delta_{M1}$	$+\delta_{M-1}$
B_{JM}/Q_J	$\frac{+iG^2(2\pi)^2}{2pq}\left(\frac{2J+1}{4\pi}\right)^{1/2}\delta_{M0}$	$-$	$\cdots$	
C_{JM}/Q_J	$\frac{+iG^2(2\pi)^2}{2Eq}\frac{m}{p}\left(\frac{2J+1}{4\pi}\right)^{1/2}\delta_{M0}$	$-$	$\cdots$	
$F_{JM}{}^0/Q_J$	$\cdots$		$\frac{-iG^2(2\pi)^2}{2Eq}\frac{m}{p}\left(\frac{2J+1}{4\pi}\right)^{1/2}\delta_{M1}$	$+\delta_{M-1}$
$G_{JM}{}^0/Q_J$	$\cdots$		$\frac{-iG^2(2\pi)^2}{2pq}\left(\frac{2J+1}{4\pi}\right)^{1/2}\delta_{M1}$	$+\delta_{M-1}$
$V_{JM}{}^+/Q_{J+1}$	$\frac{-iG^2(2\pi)^2}{2Eq}\left(\frac{2J+1}{4\pi}\right)^{1/2}\delta_{M0}$	$+$	$\cdots$	
$V_{JM}{}^-/Q_{J-1}$	$\frac{+iG^2(2\pi)^2}{2Eq}\left(\frac{J}{4\pi}\right)^{1/2}\delta_{M0}$	$-$	$\cdots$	
$U_{JM}{}^+/Q_{J+1}$	$\cdots$		$\frac{-iG^2(2\pi)^2}{2Eq}\left(\frac{J}{4\pi}\right)^{1/2}\delta_{M1}$	$+\delta_{M-1}$
$U_{JM}{}^-/Q_{J-1}$	$\cdots$		$\frac{-iG^2(2\pi)^2}{2Eq}\left(\frac{J+1}{4\pi}\right)^{1/2}\delta_{M1}$	$+\delta_{M-1}$

associated with the space parity $(-1)^{J+1}$, we find the following grouping of terms:

Singlet: $(\Psi_B{}^{1/2,1/2}-\Psi_B{}^{-1/2,-1/2})$ $B_J, C_J, V_J{}^+, V_J{}^-$;

Triplet: $(\Psi_B{}^{1/2,-1/2}-\Psi_B{}^{-1/2,1/2})$ $F_J{}^0, G_J{}^0, U_J{}^+, U_J{}^-$.

The BS equation (2) is invariant under the spin exchange operation:

$$\Psi(q_0)\rightarrow\Sigma\{\Psi(q_0)\}\equiv-\gamma_2\Psi^T(-q_0)\gamma_2. \tag{12}$$

We decompose B, C, etc., into parts even under $q_0\rightarrow -q_0$: ${}^eB$, ${}^eC$, etc., and parts odd under $q_0\rightarrow -q_0$: ${}^oB$, ${}^oC$, etc.; we then apply the operation of Eq. (12) to Eq. (6) and separate the terms that are even and odd under Σ. The result is

Singlet (odd under Σ):

$${}^eB_J,\ {}^eC_J,\ {}^eV_J{}^\pm;\ {}^oF_J{}^0,\ {}^oG_J{}^0,\ {}^oU_J{}^\pm;$$

Triplet (even under Σ):

$${}^eF_J{}^0,\ {}^eG_J{}^0,\ {}^eU_J{}^\pm;\ {}^oB_J,\ {}^oC_J,\ {}^oV_J{}^\pm.$$

Since the Born term is necessarily even under $q_0\rightarrow -q_0$, the results in the last paragraph follow also from the invariance under spin exchange (charge conjugation $\times$ parity).[16] The implication of spin-exchange invariance should perhaps be stressed: In the matrix $M^{(2)}$, the amplitudes B_J, C_J, $V_J{}^\pm$ are coupled to $F_J{}^0$, $G_J{}^0$, $V_J{}^\pm$ through elements that are odd under $q_0\rightarrow -q_0$, $k_0\rightarrow -k_0$ (Table I). The spin-exchange invariance does not imply that the odd elements of $M^{(2)}$ do not contribute, as Gourdin states,[9] but it simply states that even B_J, C_J, $V_J{}^\pm$ are coupled to odd $F_J{}^0$, $G_J{}^0$, $V_J{}^\pm$, and vice versa. The fact that the Born terms are even does not alter this conclusion, nor does the fact that odd amplitudes vanish on the mass shell ($q_0=0$). As an illustration, consider the iterative scheme starting with the even Born terms. There will be generated odd second-order terms which vanish on the mass shell, but which will contribute to even third-order terms which do not vanish on the mass shell.

The Born terms and the elements of the kernel M_{ij} exhibit a number of factors that contain kinematical branch points at $J=0$, $-\frac{1}{2}$, and -1. However, it is possible to remove these singular factors from the integral equations and the Born terms by a suitable redefinition of amplitudes to leave only poles at $J=0$,

[16] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Cambridge, Massachusetts, 1955), p. 275.

$-\frac{1}{2}$, -1, and all negative integers. Hence, these branch points need not be considered. The proper choice of amplitudes that are free of kinematical cuts is

Case I: $\lambda_1\lambda_2=\frac{1}{2},\frac{1}{2}$ or $-\frac{1}{2},-\frac{1}{2}$

$$(S_J,T_J,B_J,C_J)=(2J+1)^{1/2}(S'_J,T'_J,B'_J,C'_J)\,,$$
$$(F_J^+,G_J^+,V_J^+,U_J^+)=(J+1)^{1/2}(F'_J{}^+,G'_J{}^+,V'_J{}^+,U'_J{}^+)\,,$$
$$(F_J^-,G_J^-,V_J^-,U_J^-)=(J)^{1/2}(F'_J{}^-,G'_J{}^-,V'_J{}^-,U'_J{}^-)\,,$$
$$(F_J^0,G_J^0,V_J^0,U_J^0)=(2J+1)^{1/2}\left(\frac{J+1}{J}\right)^{1/2}\times(F'_J{}^0,G'_J{}^0,V'_J{}^0,U'_J{}^0)\,;$$

Case II: $\lambda_1\lambda_2=\frac{1}{2},-\frac{1}{2}$ or $-\frac{1}{2},\frac{1}{2}$

$$(S_J,T_J,B_J,C_J)=(2J+1)^{1/2}\left(\frac{J}{J+1}\right)^{1/2}\times(S'_J,T'_J,B'_J,C'_J)\,,$$
$$(F_J^+,G_J^+,V_J^+,U_J^+)=(J)^{1/2}(F'_J{}^+,G'_J{}^+,V'_J{}^+,U'_J{}^+)\,,$$
$$(F_J^-,G_J^-,V_J^-,U_J^-)=(J+1)^{1/2}(F'_J{}^-,G'_J{}^-,V'_J{}^-,U'_J{}^-)\,,$$
$$(F_J^0,G_J^0,V_J^0,U_J^0)=(2J+1)^{1/2}(F'_J{}^0,G'_J{}^0,V'_J{}^0,U'_J{}^0)\,.$$

We shall continue to use S, T, etc., while keeping in mind the possibility of going to the primed set.

3. COMPLEX ANGULAR MOMENTUM

The integral equation (7) can be extended to complex values of J by simply adopting the usual definition of Q_J for complex J. It must again be stressed that the equation thus extended is not the physical one for $J=0$. Attempts at applying the method of Ref. 2 are met with frustration, owing to the fact that the kernel is not square integrable (see Appendix II); while successive iterated kernels exist, their traces do not. It has been sometimes asserted[17] that no solution exists to this equation because of this singular nature of the kernel. Obviously this is an erroneous conclusion. What is true, however, is that the kernel is not square integrable and, therefore, the Fredholm alternative need not hold[5]; that is to say that, considered as a function of the coupling constant, the poles of the T matrix may not have a correspondence with the existence of a bound-state solution.

However, it is not too difficult to see that for $|E|<M$ and $\mathrm{Re}J>0$, the successive Born approximations are well defined, and for sufficiently small coupling constant the series is convergent,[18] except for the fact that as J is continued to near zero, certain of the terms become singular and integrals diverge. These are the elements containing $Q_{J-1}(y)$ which goes as J^{-1} as $J\to 0$. As was done by Sawyer under similar circumstances, we shall sum the most singular term in every order of perturbation theory into a closed form. As an inspection of Tables I and II shows, the most singular amplitudes at $J=0$ are V_J^-, U_J^-, G_J^-, and F_J^- which may be coupled among themselves. The other elements are coupled to the above singular ones through nonsingular elements of the kernel, and will be neglected.

Throughout the following analysis, we assume the weak coupling limit. A further simplification results from neglecting all remaining elements of M_{ij} which are explicitly odd in k_0. This is justified by noting that all the Born terms are even in q_0, so that the only factor preventing the complete vanishing of these terms will be proportional to

$$Q_{J-1}\left(\frac{k^2+q^2+\mu^2-k_0^2-q_0^2+2k_0q_0}{2kq}\right)-Q_{J-1}\left(\frac{k^2+q^2+\mu^2-k_0^2-q_0^2-2k_0q_0}{2kq}\right),$$

which, by the approximation we shall shortly introduce, is proportional to J, and hence less singular. Thus, the even character of the Born terms will be preserved in each iteration.

We are, thus, left with the set of equations:

$$\begin{aligned}
V_{JM}^-&=V_{B_{JM}}{}^-+\frac{iG^2}{(2\pi)^3q}\int\frac{kdkdk_0(m^2+k_0^2-E^2+k^2/2J+1)Q_{J-1}(y)V_{JM}^-}{[m^2+k^2-(k_0+E)^2][m^2+k^2-(k_0-E)^2]}\,,\\
U_{JM}^-&=U_{B_{JM}}{}^-+\frac{iG^2}{(2\pi)^3q}\int\frac{kdkdk_0(-m^2-k_0^2+E^2+k^2/2J+1)Q_{J-1}(y)U_{JM}^-}{[m^2+k^2-(k_0+E)^2][m^2+k^2-(k_0-E)^2]}\,,\\
F_{JM}^-&=F_{B_{JM}}{}^-+\frac{iG^2}{(2\pi)^3q}\int\frac{kdkdk_0\{(m^2+E^2-k_0^2-k^2/2J+1)Q_{J-1}F_{JM}^--2mEQ_{J-1}G_{JM}^-\}}{[m^2+k^2-(k_0+E)^2][m^2+k^2-(k_0-E)^2]}\,,\\
G_{JM}^-&=G_{B_{JM}}{}^-+\frac{iG^2}{(2\pi)^3q}\int\frac{kdkdk_0\{-(m^2+E^2-k_0^2+k^2/2J+1)Q_{J-1}G_{JM}^-+2mEQ_{J-1}F_{JM}^-\}}{[m^2+k^2-(k_0+E)^2][m^2+k^2-(k_0-E)^2]}\,.
\end{aligned}\qquad(14)$$

[17] As an illustration, consider the integral equation

$$\phi(x)=x^\beta+\frac{\lambda}{\pi}\int_0^\infty\frac{dy}{y+x}\phi(y)\,,\quad 0>\beta>-1\,.$$

The nth iterated kernel exists, but its trace does not. A class of eigensolutions (solutions to the homogeneous equation) is constx^α $(0>\alpha>-1)$ with the eigenvalue $\lambda=\sin\pi\alpha$. The solution is $x^\beta\sin\pi\beta/(\lambda-\sin\pi\beta)$. The pole in λ depends on the inhomogeneous term and there is no one-to-one correspondence between the pole in λ of the solution and eigenvalues.

[18] We are being mathematically cavalier at this point. The "epsilonics" could have been worked out, but the result would not have been relevant physically nor particularly illuminating mathematically.

Next we make the replacement of Q_{J-1} by its leading term in inverse powers of y. (See Sawyer[6] for a justification of this and succeeding steps.)

$$Q_{J-1}(y)=(2y)^{-J}\pi^{1/2}\Gamma(J)/\Gamma(J+\tfrac{1}{2}).$$

If we replace J by zero wherever possible, we find that $(2y)^{-J}$ cannot be set equal to 1 in the cases where the coupling term is quadratic in k or k_0, for the resulting integrals in the iteration procedure will be logarithmically divergent. As a matter of fact, Sawyer shows that the integration leads to a factor of J^{-1} in those cases. However, the terms coupling G_J^- to F_J^- are not quadratically dependent on k, and Q_{J-1} may be taken equal to simply J^{-1}. In that case we see that the cross terms contribute one less inverse power of J than the direct terms, and hence will be neglected. V_J^-, U_J^-, F_J^-, and G_J^- are decoupled in this approximation.

The Born terms may be written in the form

$$(V_B,U_B,F_B,G_B)=\frac{iG^2Q_{J-1}(y)}{q}(V_0,\cdots)$$

$$=\frac{iG^2}{qJ}\left[\frac{p^2+q^2-q_0^2+\mu^2}{pq}\right]^{-J}(V_0,\cdots),\quad(15)$$

where V_0, etc., may contain J-dependent factors. These can be removed by going to the primed amplitudes defined earlier; and therefore, they are of no consequence. The equations are now of the type discussed by Sawyer, and we shall only sketch in the solution. There are two types of equations here depending on the relative sign of k^2 and k_0^2 in the numerator of the kernel.

Consider first V_J^- and F_J^-, where k^2, k_0^2 have the same sign. Since the dominant contribution to the integrals comes from large k and k_0, the m^2 and E^2 factors are not significant, and V differs from F only in the sign of the coupling constant. Therefore, a solution for V gives one for F with that change. We have, after making Wick's rotation of the contour, neglecting E^2 compared to m^2 in the kernel, and letting $V=V'/q$,

$$V_N'=-\frac{G^2}{(2\pi)^3J}\int\frac{dk\,dk_0(m^2+k^2-k_0^2)}{(m^2+k^2+k_0^2)^2}$$

$$\times\left[\frac{qk}{q^2+k^2+\mu^2+(q_0-k_0)^2}\right]^J V'_{N-1}.\quad(16)$$

If we neglect factors odd in k_0, set μ^2 equal to zero, and make a scale transformation $(k,k_0)=m^2(k',k_0')$, we obtain

$$V'_N=\left(-\frac{2G^2}{J(2\pi)^3}\right)^{N-1}\int_0^\infty dk_1\cdots dk_{N-1}\int_0^\infty dk_{10}\cdots dk_{N-10}$$

$$\times\left\{\left(\frac{q}{m}\right)^J\left(\frac{q^2}{m^2}+k_1^2+\frac{q_0^2}{m^2}+k_{10}^2-2q_0k_0\right)^{-J}\right.$$

$$\times\frac{(1+k_1^2-k_{10}^2)}{(1+k_1^2+k_{10}^2)^2}k_1^{2J}(k_1^2+k_2^2+(k_{10}-k_{20})^2)^{-J}\cdots$$

$$\left.\times k_{N-1}^{2J}\left(k_{N-1}^2+\frac{p^2}{m^2}+k_{N-10}^2\right)^{-J}\left(\frac{p}{m}\right)^J\frac{iG^2}{J}V_0\right\}.\quad(17)$$

We next make the substitution $k_i^2=x_i$, $k_{i0}^2=y_i=t_i(1+x_i)$ and put q on the mass shell. Then we find that

$$V'_N=-i\frac{(2\pi)^3}{2}V_0\left(\frac{-2G^2}{J(2\pi)^3}\right)^N\frac{1}{2^{2N-2}}\left(\frac{p^2}{m^2}\right)^J$$

$$\times\left(\prod_i\int_0^\infty\frac{dt_i}{(t_i)^{1/2}}\int_0^\infty\frac{dx_i(1+x_i)}{(x_i)^{1/2}(1+x_i)^{1/2}}\right)$$

$$\times\left\{[x+x_1+t_1(1+x_1)]^{-J}\frac{(1+x_1)(1-t_1)}{(1+x_1)^2(1+t_1)^2}x_1^J\right.$$

$$\times[x_1+x_2+t_1(1+x_1)+t_2(1+x_2)$$

$$-2(t_1t_2(1+x_1)(1+x_2))^{1/2}]^{-J}\cdots$$

$$\left.\times x_{N-1}^J[x_{N-1}+x+t_{N-1}(1+x_{N-1})]^{-J}\right\},\quad(18)$$

where $x=p^2/m^2=q^2/m^2$. The integrals are well behaved at $x_i=0$ and the dominant contribution will come from x_i large. Therefore, $x_i\simeq(1+x_i)$, and t_i may be set equal to zero in the terms where it is added to x_i. Then we obtain the result that[19]

$$V'_N=\left[-\frac{(2\pi)^3}{2}iV_0\left(\frac{-2G^2}{J(2\pi)^3}\right)^N\frac{1}{2^{2(N-1)}}\right]$$

$$\times\left[\int_0^\infty\frac{dt(1-t)}{t^{\frac{1}{2}}(1+t)^2}\right]^{N-1}\left(\frac{p}{m}\right)^{2J}\int_0^\infty dx_1\cdots$$

$$\times dx_{N-1}(x+x_1)^{-J}(1+x_1)^{1-J}(x_1+x_2)^{-J}(1+x_2)^{1-J}\cdots$$

$$\times(1+x_{N-1})^{1-J}(x_{N-1}+x]^{-J}.\quad(19)$$

The x_i integral is given by Sawyer[6] to be equal to

$$\frac{2}{J^{N-1}}\frac{(2N-3)!}{(N-2)!N!}.$$

---

[19] The t integral here and in succeeding equations is the main difference between our equations and those of Sawyer (Ref. 6), and it is a very critical one.

The t integral, however, is found to be zero. Hence, we find that F_J^- and V_J^- do not contribute in the singular limit considered here due to cancellation between the k and k_0 integrations.

For U_J^- we have the equation

$$U'_N=-\frac{G^2}{(2\pi)^3}\frac{1}{J}\int dk dk_0\frac{(k_0^2+k^2-m^2)}{(k_0^2+k^2+m^2)^2}\times\left[\frac{qk}{q^2+k^2+\mu^2+(k_0-q_0)^2}\right]^{-J}U'_N, \quad (20)$$

where $U'=U/q$. Performing the same operations for U_N' as for V_N', we find

$$U_N=-8\pi i\left(\frac{-G^2}{J^2(2\pi)^3}\right)^N\frac{U_0}{q}\left(\frac{p^2}{m^2}\right)^J J\times\frac{(2N-3)!}{N!(N-2)!2^{2(N-1)}}\left(\int_0^\infty\frac{dt}{t^{\frac{1}{2}}(1+t)}\right)^{N-1}. \quad (21)$$

The t integral in this case is equal to π. Therefore, we have the result that

$$U_N=-8\pi^2\frac{iU_0}{q}\left(\frac{p^2}{m^2}\right)^J\times J\left\{\left(-\frac{G^2}{J^2(2\pi)^2}\right)^N\frac{(2N-3)!}{2^{2(N-1)}(N-2)!N!}\right\}. \quad (22)$$

The corresponding series may be summed to give

$$U_J^-=-\frac{8\pi^2 iU_0}{q}\left(\frac{p^2}{m^2}\right)^J\left[J-\left(J^2+\frac{G^2}{4\pi^2}\right)^{1/2}\right]\equiv-\frac{8\pi^2 iU_0}{q}U(J). \quad (23)$$

The equation for G_J^- differs from that for U_J^- only by the sign of G^2, and the corresponding sign of G_{J0}^-. Hence, we have

$$G_J^-=\frac{8\pi^2 iG_0}{q}\left(\frac{p^2}{m^2}\right)^J\left[J-\left(J^2-\frac{G^2}{4\pi}\right)^{1/2}\right]\equiv\frac{8\pi^2 iG_0}{q}G(J). \quad (24)$$

Putting in G_0 and U_0 for the four possible cases, we finally find the following solutions to our equations[20]:

[20] U_J^-, F_J^-, G_J^-, and V_J^- both in the Born amplitude and in general are what Gell-Mann calls "nonsense terms" at $J=0$ [*Proceedings of 1962 Annual International Conference on High-Energy Physics at CERN* (CERN, Geneva, 1962)].

$$\begin{aligned}G_{JM}^-(\tfrac{1}{2},\tfrac{1}{2})&=G_{JM}^-(-\tfrac{1}{2},-\tfrac{1}{2})\\&=i(2\pi)^4\frac{m}{Ep^2}\left(\frac{J}{4\pi}\right)^{1/2}G(J)\delta_{M,0},\\G_{JM}^-(\pm\tfrac{1}{2},\mp\tfrac{1}{2})&=\frac{i(2\pi)^4}{p^2}\left(\frac{J+1}{4\pi}\right)^{1/2}G(J)\delta_{M,\pm1},\\U_{JM}^-(\pm\tfrac{1}{2},\mp\tfrac{1}{2})&=-i\frac{(2\pi)^4}{Ep}\left(\frac{J+1}{4\pi}\right)^{1/2}U(J)\delta_{M,\pm1}.\end{aligned} \quad (25)$$

All other amplitudes are considered to be equal to zero.

4. HELICITY AMPLITUDES

We use here the helicity transition amplitudes defined by Goldberger *et al.*[11] The connection between the fusion amplitudes and the helicity states can be obtained from the cross sections calculated with each of them. The result is

$$\langle\lambda_1'\lambda_2'|\phi|\lambda_1\lambda_2\rangle=\frac{m}{2(2\pi)^2}\bar{U}_{\lambda_1'}(\mathbf{q})\gamma_5\Psi^{\lambda_1\lambda_2}\gamma_5 v_{-\lambda_2'}(\mathbf{q}), \quad (26)$$

where the final momentum $\mathbf{q}=p(\sin\theta,0,\cos\theta)$. After a bit of algebra we find that

$$\begin{aligned}\langle\tfrac{1}{2}\,\tfrac{1}{2}|\phi|\lambda_1\lambda_2\rangle&=-\frac{p}{(4\pi)^2}\Big[S^{\lambda_1\lambda_2}-\frac{E}{p}B^{\lambda_1\lambda_2}+\frac{m}{p}C^{\lambda_1\lambda_2}\\&\quad+\frac{\hat{q}}{p}\cdot(p\mathbf{V}^{\lambda_1\lambda_2}-E\mathbf{F}^{\lambda_1\lambda_2}+m\mathbf{G}^{\lambda_1\lambda_2})\Big],\\\langle\tfrac{1}{2}-\tfrac{1}{2}|\phi|\lambda_1\lambda_2\rangle&=-\frac{1}{(4\pi)^2}[pT^{\lambda_1\lambda_2}+mB^{\lambda_1\lambda_2}-EC^{\lambda_1\lambda_2}\\&\quad-(\mathbf{e}_1'-i\mathbf{e}_2')\cdot(p\mathbf{U}^{\lambda_1\lambda_2}+E\mathbf{G}^{\lambda_1\lambda_2}-m\mathbf{F}^{\lambda_1\lambda_2})],\end{aligned} \quad (27)$$

where we have taken $\mathbf{e}'_1$ and $\mathbf{e}'_2$ to be two unit vectors orthogonal to $\hat{q}$. Following Goldberger,[11] we define

$$\begin{aligned}\phi_1&=\langle\tfrac{1}{2}\,\tfrac{1}{2}|\phi|\tfrac{1}{2}\,\tfrac{1}{2}\rangle=\frac{1}{2p}\textstyle\sum_J(2J+1)\phi_1{}^J d_{00}{}^J(\theta),\\\phi_2&=\langle\tfrac{1}{2}\,\tfrac{1}{2}|\phi|-\tfrac{1}{2}-\tfrac{1}{2}\rangle=\frac{1}{2p}\textstyle\sum_J(2J+1)\phi_2{}^J d_{00}{}^J(\theta),\\\phi_3&=\langle\tfrac{1}{2}-\tfrac{1}{2}|\phi|\tfrac{1}{2}-\tfrac{1}{2}\rangle=\frac{1}{2p}\textstyle\sum_J(2J+1)\phi_3{}^J d_{11}{}^J(\theta),\\\phi_4&=\langle\tfrac{1}{2}-\tfrac{1}{2}|\phi|-\tfrac{1}{2}\,\tfrac{1}{2}\rangle=\frac{1}{2p}\textstyle\sum_J(2J+1)\phi_4{}^J d_{-11}{}^J(\theta),\\\phi_5&=\langle\tfrac{1}{2}\,\tfrac{1}{2}|\phi|\tfrac{1}{2}-\tfrac{1}{2}\rangle=\frac{1}{2p}\textstyle\sum_J(2J+1)\phi_5{}^J d_{10}{}^J(\theta).\end{aligned} \quad (28)$$

The ϕ_i^J can be calculated in terms of the fusion amplitudes by the formulas of Appendix IC. The final results are, putting in Eq. (25) for G_J^- and U_J^-,

$$\begin{aligned}
\phi_1^J &= -i\pi\frac{m^2}{2pE}\frac{J}{2J+1}G(J)\,,\\
\phi_2^J &= \phi_1^J\,,\\
\phi_3^J &= -\frac{i\pi}{2}\frac{J+1}{2J+1}\left[\frac{p}{E}U(J)+\frac{E}{p}G(J)\right],\\
\phi_4^J &= \frac{i\pi}{2}\frac{J+1}{2J+1}\left[\frac{p}{E}U(J)-\frac{E}{p}G(J)\right],\\
\phi_5^J &= -\frac{i\pi}{2}\frac{[J(J+1)]^{1/2}}{2J+1}\frac{m}{p}G(J)\,.
\end{aligned} \tag{29}$$

We define another set of transition amplitudes corresponding to transitions between singlet states or among the various triplet states by the equations

$$\begin{gathered}
f_0^J=\phi_1^J-\phi_2^J\,,\quad f_1^J=\phi_3^J-\phi_4^J\,,\\
f_{11}^J=\phi_1^J+\phi_2^J\,,\quad f_{22}^J=\phi_3^J+\phi_4^J\,,\\
f_{12}^J=2\phi_5^J\,,
\end{gathered} \tag{30}$$

and

$$\begin{gathered}
f_1=E(\phi_1-\phi_2)\,,\quad f_3=E\left(\frac{\phi_3}{1+z}-\frac{\phi_4}{1-z}\right),\\
f_2=E(\phi_1+\phi_2)\,,\quad f_4=E\left(\frac{\phi_3}{1+z}+\frac{\phi_4}{1-z}\right),\\
f_5=\frac{-2m}{\sin\theta}\,,
\end{gathered} \tag{31}$$

where $z=\cos\theta$. The f_i have a convenient partial-wave expansion in terms of Legendre polynomials.[12] This is

$$\begin{aligned}
f_1&=\frac{E}{2p}\sum_{J=0}^{\infty}(2J+1)f_0^JP_J(z),\\
f_2&=\frac{E}{2p}\sum_{J=0}^{\infty}(2J+1)f_{11}^JP_J(z)\,,\\
f_3&=\frac{E}{2p}\sum_{J=1}^{\infty}\frac{(2J+1)}{J(J+1)}\{f_1^J[zP_J'(z)]'-f_{22}^JP_J''(z)\}\,,\\
f_4&=\frac{E}{2p}\sum_{J=1}^{\infty}\frac{(2J+1)}{J(J+1)}\{f_{22}^J[zP_J'(z)]'-f_1^JP_J''(z)\}\,,\\
f_5&=-\frac{m}{2p}\sum_{J=1}^{\infty}\frac{(2J+1)}{[J(J+1)]^{1/2}}f_{12}^JP'_J(z)\,.
\end{aligned} \tag{32}$$

The primes on the P_J indicate differentiation. The f_i^J may now be written, by means of the Sommerfeld-Watson transformation,[12] as an integral over the angular momentum. The contour may be opened up in the usual manner and pushed to within an ϵ of $\mathrm{Re}J=0$, except that we must take into account any singularities of the partial-wave amplitudes; these will be the singularities of the ϕ_i^J, which by inspection of Eq. (29) are seen to be cuts running from $-G/2\pi$ to $G/2\pi$ and from $-iG/2\pi$ to $iG/2\pi$. Thus, our contour becomes that indicated in Fig. 3, where the dashed line refers to the original contour and the solid line is the opened contour.

It should be pointed out that the contour, before deformation, starts at $J=1$, even for f_1 and f_2. We isolate the $J=0$ contributions to these amplitudes since our derived results are not valid at that point. In as much as we ultimately are going to look at the large z limits (Sec. 5), the contribution of these $J=0$ terms may be neglected since they are independent of z. The domain of analyticity in the complex J plane is seen to be all J with $\mathrm{Re}J>0$, except for a cut along the real axis extending to $J=G/2\pi$. This cut, being the singularity furthest to the right, will give the dominant contribution to the high z behavior. If the discontinuity across the cut is represented by $\Delta f_0^J=f_0^{J+}-f_0^{J-}$, etc., the leading behavior of the f_i will be given by the integrals

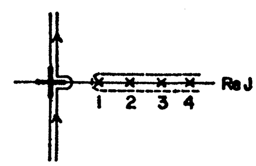

FIG. 3. Integration contours in the complex J plane for the f_i.

$$\begin{aligned}
f_1&=\frac{E}{4ip}\int_\epsilon^{G/2\pi}\frac{2J+1}{\sin\pi J}P_J(-z)\Delta f_0^J\,,\\
f_2&=\frac{E}{4ip}\int_\epsilon^{G/2\pi}\frac{2J+1}{\sin\pi J}P_J(-z)\Delta f_{11}^J\,,\\
f_3&=\frac{-E}{4ip}\int_\epsilon^{G/2\pi}\frac{2J+1}{J(J+1)}\frac{[-zP'_J(-z)]'}{\sin\pi J}\Delta f_{22}^J\,,\\
f_4&=\frac{-E}{4ip}\int_\epsilon^{G/2\pi}\frac{2J+1}{J(J+1)}\frac{[-zP'_J(-z)]'}{\sin\pi J}\Delta f_1^J\,,\\
f_5&=\frac{m}{4ip}\int_\epsilon^{G/2\pi}\frac{2J+1}{[J(J+1)]^{1/2}}\frac{P'_J(-z)}{\sin\pi J}\Delta f_{12}^J\,.
\end{aligned} \tag{33}$$

Only the term proportional to $[zP'_J]'$ is retained in f_3 and f_4 since the leading term of P''_J is down by a factor of z and we desire only the asymptotic dependence. The discontinuities in the partial-wave amplitudes are found to be

$$\Delta f_0{}^J=0\,,$$

$$\Delta f_{11}{}^J=-2\pi\frac{m^2}{Ep}\frac{J}{2J+1}f(J)\,,$$

$$\Delta f_{22}{}^J=-2\pi\frac{E}{p}\frac{J+1}{2J+1}f(J)\,, \tag{34}$$

$$\Delta f_1{}^J=0\,,$$

$$\Delta f_{12}{}^J=-2\pi\frac{m}{E}\frac{[J(J+1)]^{1/2}}{2J+1}f(J)\,,$$

where

$$f(J)=(p^2/m^2)^J(G^2/4\pi^2-J^2)^{1/2}.$$

Then, letting $P_J(-z)=(-z)^J$, $P'_J(-z)=J(-z)^{J-1}$, $[-zP'_J(-z)]'(-z)]'=J^2(-z)^{J-1}$, and using the result of Appendix III, we get the results

$$f_1=0\,,\qquad f_3=0\,,$$

$$f_2=\frac{im^2}{4p^2}\frac{G^{\frac{1}{2}}(-t/m^2)^{G/2\pi}}{[\ln(-t/m^2)]^{3/2}}\equiv\frac{im^2}{4p^2}A\,,\quad f_4=\frac{iE^2}{2t}A\,,$$

$$f_5=\frac{-im^2}{2t}A\,, \tag{35}$$

where 2^J has been taken equal to one. Hereafter, since no confusion can arise, we will assume all energies to be measured in terms of the nucleon mass, and take $m^2=1$.

5. INVARIANT AMPLITUDES

In the case of nucleon-antinucleon scattering one can show on the basis of parity, charge conjugation, and time reversal invariance that five amplitudes are sufficient to describe the scattering process. We define a transition matrix T_{fi} in such a way that[20]

$$T_{fi}=\bar{u}_\alpha{}^{\lambda_1'}(\mathbf{q})\bar{v}_\beta{}^{-\lambda_2}(\mathbf{p})M_{\alpha\beta;\delta\epsilon}u_\delta{}^{\lambda_1}(\mathbf{p})v_\epsilon{}^{-\lambda_2'}(\mathbf{p})$$
$$=4\pi E\langle\lambda_1'\lambda_2'|\phi|\lambda_1\lambda_2\rangle\,, \tag{36}$$

where

$$M_{\alpha\beta;\delta\epsilon}=a_1\delta_{\alpha\delta}\delta_{\beta\epsilon}+a_2[\gamma_{\alpha\delta}\cdot K\delta_{\beta\epsilon}+\delta_{\alpha\delta}\gamma_{\beta\epsilon}\cdot P]$$
$$+a_3\gamma_{\alpha\delta}\cdot K\gamma_{\beta\epsilon}\cdot P+a_4\gamma_{\alpha\delta5}\gamma_{\beta\epsilon5}$$
$$+a_5\gamma_{\alpha\tau5}\gamma_{\tau\delta}\cdot K\gamma_{\beta\sigma}\cdot P\gamma_{\sigma\epsilon5}\,, \tag{37}$$

where $P=\frac{1}{2}(p+p')$ and $K=\frac{1}{2}(k+k')$. By computing T_{fi} explicitly for the five helicity combinations involved in the ϕ_i, we may relate the a_i to the ϕ_i in the center-of-mass system. The spinors used earlier in relating the fusion amplitude to the ϕ_i are to be used again here. A tedious calculation leads to the following relations:

$$4\pi E\phi_1=[-a_1+2(E^2+p^2)a_2-(E^2+p^2)^2a_3$$
$$-4E^2p^2a_5]\tfrac{1}{2}(1+\cos\theta)\,,$$
$$4\pi E\phi_2=[E^2a_1-2E^2a_2+E^2a_3-p^2a_4]\tfrac{1}{2}(1-\cos\theta)\,,$$
$$4\pi E\phi_3=[-a_1+2(E^2+p^2)a_2-(E^2+p^2)^2a_3$$
$$+4E^2p^2a_5]\tfrac{1}{2}(1+\cos\theta), \tag{38}$$
$$4\pi E\phi_4=[-E^2a_1+2E^2a_2-E^2a_3-p^2a_4]\tfrac{1}{2}(1-\cos\theta)\,,$$
$$4\pi E\phi_5=[Ea_1-2E^3a_2+E(E^2+p^2)a_3]\tfrac{1}{2}(\sin\theta)\,.$$

If one expresses the ϕ_i in terms of the f_i, and uses the fact that f_1 and f_3 are zero, one can easily solve for the a_i in terms of the f_i. We neglect one compared to z, and E compared to one to simplify matters. $(p^2=-1)$

$$a_1=-(\pi/2E^2)\left(f_4+\frac{f_2}{z}+2E^2f_5\right),$$
$$a_2=(\pi/2E^2)\left(f_4+\frac{f_2}{z}+4E^4f_5\right),$$
$$a_3=(\pi/2E^2)(4E^2f_5-f_2/z-f_4)\,, \tag{39}$$
$$a_4=2\pi(f_4-f_2/z)\,,$$
$$a_5=-(\pi/2E^2)(f_4-f_2/z)\,.$$

Putting in the values of the f_i, we obtain $(z^{-1}=2p^2/t$ and $s=4E^2)$

$$a_1=-(i\pi/st)A\,,\quad a_3=-(i\pi/st)A\,,$$
$$a_2=(i\pi/st)A\,,\quad a_4=-(i\pi/t)A\,, \tag{40}$$
$$a_5=(i\pi/st)A\,.$$

6. HIGH-ENERGY BEHAVIOR IN THE CROSSED CHANNEL

We are now able to calculate the high-energy contribution of the annihilation-creation process to the differential cross section in the forward direction (small s).

$$\frac{d\sigma}{d\Omega}=\frac{1}{(2\pi)^2}\frac{1}{t}|T_{fi}|^2\,,$$

where a sum over initial and average over final spin states is implied. This gives, by a standard trace calculation,

$$\frac{d\sigma}{d\Omega}=\frac{t}{16(2\pi)^2}\{|a_1|^2+4|a_2|^2+|a_3|^2+|a_4|^2+s^2|a_5|^2$$
$$+2(a_1{}^*a_2+a_2{}^*a_1)+(a_1{}^*a_3+a_3{}^*a_1)+2(a_2{}^*a_3+a_3{}^*a_2)\}\,.$$

or, using Eq. (40) and putting in A from Eq. (35),

$$d\sigma/d\Omega=G(t)^{(G/\pi)-1}/32(\ln t)^3\,. \tag{41}$$

This cross section, while interesting in the manner in which it exhibits the effect of the Regge cut, has little meaning in itself since the meson exchange diagrams would be expected to dominate at high energies. However, we can obtain the total cross section for nucleon-antinucleon annihilation at high energies by the use of the optical theorem. Specifically, we find by a comparison of Nth-order terms that

$$\sigma_{\rm ann}=-2\pi^2\Big(\tfrac{1}{4}\sum_{\lambda_1\lambda_2}{\rm Im}R^t(s=0)\Big),\qquad(42)$$

where $R^t(s=0)$ is the R matrix in the t channel in the forward direction $(s=0)$ with $\lambda_1=\lambda_1'$, $\lambda_2=\lambda_2'$. R^t is given by

$$R^t=\frac{4}{(2\pi)^2t}T_{fi}.$$

By means of another simple trace calculation we find

$$\sum_{\lambda_1\lambda_2}R^t(s=0)=\frac{2}{(2\pi)^2}[a_1+2a_2+a_3+a_4].$$

Putting in the values of the a_i from Eq. (40), and letting $-1=e^{-i\pi}$ we see that

$$\sum_{\lambda_1\lambda_2}R^t(s=0)=\frac{-i}{2\pi}\frac{G^{1/2}}{t}\frac{e^{-iG/2}(t)^{G/2\pi}}{[\ln t-i\pi]^{3/2}}.\qquad(43)$$

Under our convention in the use of factors of i, the ${\rm Im}R$ referred to above is actually half the discontinuity of R across the cut going from $t=0$ to ∞. The contribution from the denominator is neglected since it is down by a factor of $[\ln t]^{-1}$. Hence, we have the results

$${\rm Im}\sum R=-\frac{1}{4\pi t}\left[\frac{G}{\ln t}\right]^{3/2}(t)^{G/2\pi}$$

and

$$\sigma_{\rm ann}=\frac{\pi}{8}\left[\frac{G}{\ln t}\right]^{3/2}(t)^{G/2\pi-1}.\qquad(44)$$

We have shown an example of the occurrence of a "Regge cut" in field theory in this paper. The argument presented is valid only in the weak coupling limit, and it is quite possible that in a strong coupling, the leading singularity in the J plane is a pole in the model considered. In fact, even the weak coupling results might be considerably altered by considering diagrams with crossed meson lines or nucleon annihilation in addition to the ladder diagrams used. Interference effects could lead to a cancellation of the cut and leave a Regge pole as the rightmost singularity. However, the mathematics is then prohibitive.

We have considered quantum electrodynamics with photons of finite mass. The result is very similar to the $ps-ps$ theory considered here[21]; in the limit as the photon mass goes to zero, our result, however, loses its validity, since the approximation[22] used in Sec. 3 holds good only for finite boson mass μ. In quantum electrodynamics, the electron positron annihilation into n photons is dominated by the process in which two energetic photons and $n-2$ soft photons are produced.[23] The approximation made in Sec. 3 is valid only for $\mu\neq 0$, and precludes the soft-photon contributions in the limit $\mu\to 0$.

ACKNOWLEDGMENT

We wish to thank Professor R. F. Sawyer for communication of his results prior to publication and numerous discussions on the subject.

APPENDIX I

A. Relations with Vector Spherical Harmonics

Let $\hat{q}$ be a unit vector in the θ, ϕ direction, and we have[24]

$$\hat{q}Y_{JM}=-\left(\frac{J+1}{2J+1}\right)^{1/2}\mathbf{Y}_{J\,J+1\,M}+\left(\frac{J}{2J+1}\right)^{1/2}\mathbf{Y}_{J\,J-1\,M},\qquad(\text{A1})$$

$$\hat{q}\cdot\mathbf{Y}_{JJM}=0,$$

$$\hat{q}\cdot\mathbf{Y}_{JJ+1M}=-\left(\frac{J+1}{2J+1}\right)^{1/2}Y_{JM},\qquad(\text{A2})$$

$$\hat{q}\cdot\mathbf{Y}_{JJ-1M}=\left(\frac{J}{2J+1}\right)^{1/2}Y_{JM},$$

$$\hat{q}\times\mathbf{Y}_{JJ+1M}=i\left(\frac{J}{2J+1}\right)^{1/2}\mathbf{Y}_{JJM},\qquad(\text{A3})$$

$$\hat{q}\times\mathbf{Y}_{JJ-1M}=i\left(\frac{J+1}{2J+1}\right)^{1/2}\mathbf{Y}_{JJM},$$

$$\hat{q}\times\mathbf{Y}_{JJM}=i\left[\left(\frac{J}{2J+1}\right)^{1/2}\mathbf{Y}_{JJ+1M}+\left(\frac{J+1}{2J+1}\right)^{1/2}\mathbf{Y}_{JJ-1M}\right].\qquad(\text{A4})$$

[21] A. R. Swift and B. W. Lee (unpublished).

[22] The approximation

$$Q_J(y)\to\pi^{1/2}\frac{\Gamma(J+1)}{\Gamma(J+\frac{3}{2})}(2y)^{-J-1},$$

where

$$y=[k^2+k'^2+\mu^2+(k_0-k'_0)^2]/2kk'$$

is valid for large y. It becomes a bad one when $y\to 1$. This happens only when $k=k'\to\infty$ for $\mu\neq 0$; but happens for all values of $k=k'$ for $\mu=0$.

[23] S. N. Gupta, Phys. Rev. **98**, 1502 (1955); D. R. Yennie, S. C. Frautschi, and H. Suura, Ann. Phys. (N.Y.) **13**, 379 (1961); J. Joseph, Phys. Rev. **103**, 481 (1956); K. G. Mahanthappa, thesis, Harvard University, 1961 (unpublished).

[24] A. Akhiezer and V. B. Berestetsky, *Quantum Electrodynamics* (Technical Information Service Extension, Oak Ridge, Tennessee, 1953), Part I, p. 32.

B. Further Relations with Vector Spherical Harmonics

Choosing $\mathbf{e}_0$, $\mathbf{e}_1$, and $\mathbf{e}_2$ to belong to an external coordinate system, we easily obtain the following relations from the basic definition of the $\mathbf{Y}_{JLM}$:

$$\mathbf{e}_0 Y_{L0} = \left(\frac{L+1}{2L+1}\right)^{1/2} \mathbf{Y}_{L+1\,L0} - \left(\frac{L}{2L+1}\right)^{1/2} \mathbf{Y}_{L-1\,L0}, \quad \text{(A5)}$$

$$(\mathbf{e}_1 + i\mathbf{e}_2) Y_{L0} = -\left(\frac{L+2}{2L+1}\right)^{1/2} \mathbf{Y}_{L+1\,Li} + \mathbf{Y}_{LL1} - \left(\frac{L-1}{2L+1}\right)^{1/2} \mathbf{Y}_{L-1\,L1}, \quad \text{(A6)}$$

$$(\mathbf{e}_i - i\mathbf{e}_2) Y_{L0} = \left(\frac{L+2}{2L+1}\right)^{1/2} \mathbf{Y}_{L+1\,L-1} + \mathbf{Y}_{L\,L-1} + \left(\frac{L-1}{2L+1}\right)^{\frac{1}{2}} Y_{L-1\,L-1}. \quad \text{(A7)}$$

C. The $d_{mn}{}^J(\theta)$ Functions

Using Eq. (A2) and the relation $Y_{JM}(\theta,\phi=0) = [(2J+1)/4\pi]^{1/2} d_{M0}{}^J(\theta)$ we see immediately that

$$\hat{q}\cdot\mathbf{Y}_{JJ-10} = (J/4\pi)^{1/2} d_{00}{}^J(\theta), \quad \text{(A8)}$$

$$\hat{q}\cdot\mathbf{Y}_{JJ-11} = (J/4\pi)^{1/2} d_{10}{}^J(\theta). \quad \text{(A9)}$$

Furthermore, we can express $\mathbf{e}'_1 - i\mathbf{e}'_2$, as defined just below Eq. (27), in the form

$$\mathbf{e}'_1 - i\mathbf{e}'_2 = \sqrt{2} \textstyle\sum_q d_{1q}{}'(\theta)\mathbf{e}_{-q},$$

so that

$$(\mathbf{e}'_1 - i\mathbf{e}'_2)\cdot\mathbf{Y}_{JLM} = \sum_q C_{L1}(J,M;M-q,q)\left(\frac{2J+1}{4\pi}\right)^{1/2} d_{M-q0}{}^J d_{1q}{}'(-1)^q$$
$$= (-1)^M \sqrt{2} C_{L1}(J,1;01) d_{1M}{}^J \left(\frac{2J+1}{4\pi}\right)^{1/2}. \quad \text{(A10)}$$

The last step is a consequence of the addition theorem[25] for the $d_{mn}{}^J$ functions and the orthonormality of the Clebsch-Gordan coefficients.

APPENDIX II

Integrability of the Bethe-Salpeter Kernel

Taking a representative kernel from our integral equations to be proportional to k^2 and $Q_J(y)$, we consider the following integral

$$\int dk_1 dk_2 dk_{0_1} dk_{0_2} |K|^2 = \int_{-\infty}^{\infty} dk_{10} dk_{20} \int_0^{\infty} dk_1 dk_2 \times \frac{k_1^2 k_2^2}{[k_1^2+k_{10}^2+m^2]^2 [k_2^2+k_{20}^2+m^2]^2} \times \left| Q_J\left(\frac{k_1^2+k_2^2+(k_{10}-k_{20})^2+\mu^2}{2k_1k_2}\right)\right|^2.$$

By inspection, one can easily see that the most divergent part of the integral will come from the region where k_1 and k_2 are both large; otherwise $Q_J(y)$ will provide sufficient damping to insure the convergence of the integral. m^2 can be set equal to one by a scale transformation on the k_i and k_{i0}; μ^2 can be neglected. If we make the change of variables, $k_{i0} = t_i(1+x_i)^{1/2}$ and $k_i^2 = x_i$, we find the result that the integral becomes

$$\int_{-\infty}^{\infty} \frac{dt_1 dt_2}{(1+t_1^2)^2(1+t_2^2)^2} \int_0^{\infty} dx_1 dx_2 \left(\frac{1+x_1}{x_1}\right)^{1/2} \left(\frac{1+x_2}{x_2}\right)^{1/2} \frac{x_1}{(1+x_1)^2} \frac{x_2}{(1+x_2)^2} \times \left| Q_J\left(\frac{x_1+x_2+t_1^2(1+x_1)+t_2^2(1+x_2)-2t_1t_2(1+x_1)^{1/2}(1+x_2)^{1/2}}{2(x_1x_2)^{1/2}}\right)\right|^2.$$

In the limit of large x_i, we can let $x_i = 1 + x_i$. Then, if $x_2 = \lambda x_1$, we obtain the integral in the form

$$\int_{-\infty}^{\infty} \frac{dt_1 dt_2}{(1+t_1^2)^2(1+t_2^2)^2} \int_0^{\infty} d\lambda \int_0^{\infty} \frac{x_1 dx_1}{[1+x_1][1+\lambda x_1]} \times \left| Q_J\left(\frac{1+\lambda+t_1^2+\lambda t_2^2-2t_1t_2\sqrt{\lambda}}{2\sqrt{\lambda}}\right)\right|^2.$$

The x_1 integral can easily be done and is found to diverge logarithmically. Thus, we see that our kernel is not square integrable, independent on the value of J; therefore, there is no region of the complex J plane in which Fredholm theory would be applicable.

APPENDIX III

We calculate the asymptotic value of the integral arising in Eq. (35) in the following fashion. Consider

[25] Ref. 15, p. 61.

the integral

$$I=\int_0^a dx\, e^{xb}(a^2+x^2)^{1/2},$$

where $a\ll 1$, $b\to\infty$. Let $y=bx$, and $(a+x)^{1/2}=(2a)^{1/2}$. Then we find

$$I=\left(\frac{2a}{b^3}\right)^{1/2}\int_0^{ab} dy\, e^y(ab-y)^{1/2}=\left(\frac{a\pi}{2b^3}\right)^{1/2} e^{ab},$$

where in the last step we let $ab\to\infty$.

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Divergence and Summability of the Many-Fermion Perturbation Series*

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We investigate the convergence of the many-fermion perturbation series and show, for the case of the square-well potential, that it is a divergent series. We bound the rate of divergence and show that, by using appropriate summation procedures, it may be summed to the physically correct sum, provided the density is low enough.

1. INTRODUCTION

THERE is a widely held view[1] that the many-fermion perturbation theory as currently formulated is "sufficiently established on theoretical grounds." The purpose of this paper is to question that view. For the sake of explicitness we will consider a system of spinless fermions interacting via a square-well potential. We first establish, in the second section, that the radius of convergence for the ground-state energy of the N-body system (at fixed density) tends to zero as N tends to infinity at least as fast as $N^{-\gamma}$, where γ is any positive number less than $\frac{1}{3}$. This result implies that the perturbation series is, at best, an asymptotic one.

In the third section we consider the complete perturbation series and bound every order. We find that it diverges no faster than a geometrical series times $(n!)$, where n is the order of the term. We also give an argument based on the BCS theory of superconductivity that, in general, the series diverges at least this fast. In the final section we consider the problem of assigning a meaning to the sum of the series and show, provided the density is low enough (small compared to the jamming density for hard spheres), that it may be summed, even though divergent, to the $\lim_{N\to\infty}E_N(V)$, where $E_N(V)$ is the energy per particle for a potential of real, positive strength V in the N-body problem. We advance some arguments to support the conjecture that the methods we present give the physically correct sum in general when the physical system has no long-range order.

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[1] See, for instance, H. A. Bethe, B. H. Brandow, and A. G. Petschek, Phys. Rev. **129**, 225 (1963).

2. THE DIVERGENCE OF THE PERTURBATION SERIES

In this section we shall establish that the many-body perturbation series is, at best, an asymptotic series and not a convergent one, and estimate approximately the angular region in which it is asymptotic. The first important point is, that as the number of particles N tends to infinity, each order in the Rayleigh-Schrödinger perturbation series for E/N, energy per particle, tends to a finite limit. This was first asserted by Brueckner[2] and later proved by Goldstone.[3] The second important point, which we will discuss below, is that in the limit as N tends to infinity there occur branch points in the energy which move to the origin of the complex potential V plane.

The analysis of Cooper[4] for a simple model without kinetic energy may not be germane as it seems that he proves that the energy expansion has zero radius (or infinite in special cases) of convergence even for two particles in a box. This result is not appropriate to ordinary perturbation theory with a kinetic energy present.

In order to investigate the many-body problem with a square-well interaction, we shall first investigate the problem of a particle in a spherical box with a square-well potential of strength V near the origin. The potential is

$$\begin{aligned} &V, \quad 0<r<a,\\ &0, \quad a<r<a+b,\\ &+\infty, \quad a+b<r. \end{aligned} \tag{2.1}$$

[2] K. A. Brueckner and C. A. Levinson, Phys. Rev. **97**, 1344 (1955). See also H. A. Bethe, *ibid.* **103**, 1353 (1956) for an extensive list of references.

[3] J. Goldstone, Proc. Roy. Soc. (London) **A239**, 267 (1957).

[4] L. N. Cooper, Phys. Rev. **122**, 1021 (1961).